NACA

RESEARCH MEMORANDUM

DETERMINATION OF COUPLED MODES AND FREQUENCIES OF SWEPT WINGS BY USE OF POWER SERIES

By

Roger A. Anderson

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
WASHINGTON
October 20, 1947

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

RESEARCH MEMORANDUM

DETERMINATION OF COUPLED MODES AND FREQUENCIES OF SWEPT WINGS BY USE OF POWER SERIES

By Roger A. Anderson

SUMMARY

A solution is presented for the coupled modes and frequencies of swept wings mounted on a fuselage. The energy method is used in conjunction with power series to obtain the characteristic equations for both symmetrical and antisymmetrical vibration. A numerical example which is susceptible to exact solution is presented, and the results for the exact solution and the solution presented in this paper show excellent agreement.

INTRODUCTION

Except for certain idealized cases, the natural vibration modes and frequencies of wings (swept or unswept) cannot be found by exact analysis, thus making it necessary to resort to approximate methods of solution. This paper presents such a solution for the symmetrical and antisymmetrical mass coupled bending and torsional modes and frequencies of a nonuniform swept wing mounted on a fuselage. The energy method is used to derive two sets of linear characteristic equations; one for symmetrical and the other for antisymmetrical vibrations. The analysis assumes that the swept wings are essentially beams and that the deflection and rotation of the beams conform to standard engineering beam theory. Such an analysis may or may not be strictly applicable to wings having a large root chord (especially when combined with appreciable sweep) because the distortions in the vicinity of the root are not fully understood.

The important feature of the method presented herein is the simplification that results from the choice of simple power series for the expansion of the deflection and rotation of the vibrating wing. Comparison of results from a power series solution to an exact solution for the modes and frequencies of an idealized structure shows that only a few terms are needed in the expansions to obtain good accuracy.

SYMBOLS

L	length of semispan along elastic axis
E	Young's modulus of elasticity
G	modulus of elasticity in shear
I	bending moment of inertia of cross sections perpendicular to elastic axis of wing
I_m	mass polar moment of inertia per unit length of wing about elastic axis
J	torsion constant for cross sections perpendicular to elastic axis of wing
I_{FP}	one-half of pitching polar moment of inertia of fuselage about elastic axis of wing
I_{FR}	one-half of rolling polar moment of inertia of fuselage about its longitudinal axis
y	coordinate denoting deflection of elastic axis of wing
ϕ	coordinate denoting twist of wing about elastic axis
x	coordinate denoting distance along elastic axis measured from center line of fuselage or root of wing
Λ	angle of sweep, measured between wing elastic axis and line perpendicular to fuselage
m	mass of wing per unit length (w/g)
w	weight per unit length
g	acceleration due to gravity
m_F	one-half of mass of fuselage
ω	circular frequency of natural mode of vibration, radians per second

e distance between mass center of wing cross sections or points of mass concentration and elastic axis of wing; positive when mass center lies forward of elastic axis

e_F distance between mass center of fuselage and wing elastic axis; positive when mass center lies forward of elastic axis

a_n coefficient of nth term in power series expansion for y

b_n coefficient of nth term in power series expansion for ϕ

n, i, j integers (1, 2, 3, . . .)

λ distance between equally spaced spanwise stations

ENERGY EXPRESSIONS AND DEFLECTION FUNCTIONS

To determine the modes of vibration of wings it is sufficient to consider the equilibrium of the semispan only. The airplane is divided along its longitudinal axis with a coordinate system assigned as shown in figure 1. In this paper, the fuselage is assumed inflexible and it therefore possesses only the rigid body properties.

For vibration of this system, the energies considered are the bending, twisting, and kinetic energies of the wing semispan and half the kinetic energy of the fuselage. At maximum displacement of the wing the sum of the strain energy of bending and strain energy of twisting is given by the well-known expression,

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^L GJ\left(\frac{d\phi}{dx}\right)^2 dx \tag{1}$$

The kinetic energy of the wing as it passes through the equilibrium position is given by (see appendix for derivation),

$$V_1 = \frac{\omega^2}{2}\int_0^L my^2\ dx + \omega^2\int_0^L me\phi y\ dx + \frac{\omega^2}{2}\int_0^L I_m\phi^2\ dx \tag{2}$$

Half of the kinetic energy of the fuselage is given similarly by (see appendix for derivation),

$$V_2 = \frac{\omega^2}{2}\left(m_F y^2 + 2m_F e_F \Theta y + I_{FP}\Theta^2 + I_{FR}\Psi^2\right)_{x=0} \tag{3}$$

where

Θ angle of pitch of fuselage

Ψ angle of roll of fuselage

It can be shown by geometry that these angles are related to the angle of twist at the root, the slope of the elastic axis at the root, and the angle of sweep by the following relations:

$$\Theta = \left(\phi \cos \Lambda - \frac{dy}{dx} \sin \Lambda\right)_{x=0} \tag{4}$$

$$\Psi = \left(\phi \sin \Lambda + \frac{dy}{dx} \cos \Lambda\right)_{x=0} \tag{5}$$

By the energy method, functions are chosen to represent the deflection and twist of the elastic axis of the wing. It is convenient to represent the deflection and twist by the two general power series

$$y = a_0 + a_1\left(\frac{x}{L}\right) + a_2\left(\frac{x}{L}\right)^2 + \ldots + a_n\left(\frac{x}{L}\right)^n + \ldots \tag{6}$$

$$\phi = b_0 + b_1\left(\frac{x}{L}\right) + b_2\left(\frac{x}{L}\right)^2 + \ldots + b_n\left(\frac{x}{L}\right)^n + \ldots \tag{7}$$

With these series, the geometrical boundary conditions at the wing root $\left(\frac{x}{L} = 0\right)$ can be satisfied for both the symmetrical and antisymmetrical modes of vibration through use of simple relationships between the coefficients a_1 and b_0. These relationships will be given later. The use of power series also allows for complete freedom in choice of deflection and twist along the wing.

BOUNDARY CONDITIONS AND CHARACTERISTIC EQUATIONS

On substitution of equations (4), (5), (6), and (7) into equations (1), (2), and (3), the energies U, V_1, and V_2 will be expressed in terms of the unknown coefficients a_n, b_n and the unknown frequency ω. It is convenient at this point to introduce the boundary conditions.

For symmetrical vibration, the constraining relation at the wing root is that the fuselage shall not roll, or

$$\psi = \left(\phi \sin \Lambda + \frac{dy}{dx} \cos \Lambda\right)_{\frac{x}{L}=0} = 0 \tag{8}$$

which gives the following simple relation between a_1 and b_0

$$a_1 = -b_0 L \tan \Lambda \tag{9}$$

Elimination of a_1 from V_1 and V_2 (a_1 does not appear in U) by means of this relation leads to the solution for symmetrical modes and frequencies.

For antisymmetrical vibrations, the constraining relations at the wing root are that the deflection is zero and that the fuselage shall not pitch, or

$$(y)_{\frac{x}{L}=0} = 0 \tag{10}$$

and

$$\Theta = \left(\phi \cos \Lambda - \frac{dy}{dx} \sin \Lambda\right)_{\frac{x}{L}=0} = 0 \tag{11}$$

which give the following relations

$$a_o = 0 \tag{12}$$

$$b_o = \frac{a_1}{L} \tan \Lambda \tag{13}$$

Substitution of these relations into the expressions for V_1 and V_2 (a_o and b_o do not appear in U) leads to a solution for the antisymmetrical modes and frequencies.

The characteristic modes and frequencies of vibration (symmetrical and antisymmetrical) can be found by minimization of the expression $U - V_1 - V_2$ with respect to the unknown coefficients a_i and b_i. The following sets of linear homogeneous equations are derived in this way for the two types of vibration.

<u>Symmetrical vibration</u>.- For a_i

$$a_o\left(A_o + m_F\right) + \sum_{n=2}^{s} a_n A_n + b_o\left(B_o + \frac{m_F e_F}{\cos \Lambda} - LA_1 \tan \Lambda\right) + \sum_{n=1}^{t} b_n B_n = 0$$

$$(i = 0) \tag{14}$$

$$a_o A_i + \sum_{n=2}^{s} a_n\left(A_{i+n} - \frac{C_{i+n}}{\omega^2}\right) + b_o\left(B_i - LA_{i+1} \tan \Lambda\right) + \sum_{n=1}^{t} b_n B_{i+n} = 0$$

$$(i = 2, 3, 4, \ldots) \tag{15}$$

For b_i

$$a_o\left(B_o + \frac{m_F e_F}{\cos\Lambda} - LA_1 \tan\Lambda\right) + \sum_{n=2}^{s} a_n\left(B_n - LA_{n+1} \tan\Lambda\right)$$

$$+ b_o\left(D_o + L^2 A_2 \tan^2\Lambda + \frac{I_{FP}}{\cos^2\Lambda} - 2LB_1 \tan\Lambda\right)$$

$$+ \sum_{n=1}^{t} b_n\left(D_n - LB_{n+1} \tan\Lambda\right) = 0$$

$$(i = 0) \tag{16}$$

$$a_o B_i + \sum_{n=2}^{s} a_n B_{i+n} + b_o\left(D_i - LB_{i+1} \tan\Lambda\right) + \sum_{n=1}^{t} b_n\left(D_{i+n} - \frac{E_{i+n}}{\omega^2}\right) = 0$$

$$(i = 1, 2, 3, \ldots) \tag{17}$$

Antisymmetrical vibration.- For a_i

$$a_1\left(A_2 + \frac{1}{L^2}D_o \tan^2\Lambda + \frac{I_{FR}}{L^2 \cos^2\Lambda} + \frac{2}{L}B_1 \tan\Lambda\right)$$

$$+ \sum_{n=2}^{s} a_n\left(A_{n+1} + \frac{1}{L}B_n \tan\Lambda\right) + \sum_{n=1}^{t} b_n\left(\frac{1}{L}D_n \tan\Lambda + B_{n+1}\right) = 0$$

$$(i = 1) \tag{18}$$

$$a_1\left(A_{i+1} + \frac{1}{L}B_i \tan\Lambda\right) + \sum_{n=2}^{s} a_n\left(A_{i+n} - \frac{C_{i+n}}{\omega^2}\right) + \sum_{n=1}^{t} b_n B_{i+n} = 0$$

$$(i = 2, 3, 4, \ldots) \tag{19}$$

For b_i

$$a_1\left(\frac{1}{L}D_i \tan\Lambda + B_{i+1}\right) + \sum_{n=2}^{s} a_n B_{i+n} + \sum_{n=1}^{t} b_n\left(D_{i+n} - \frac{E_{i+n}}{\omega^2}\right) = 0$$

$$(i = 1, 2, 3, \ldots) \tag{20}$$

The constants A_{i+n}, B_{i+n}, C_{i+n}, D_{i+n}, and E_{i+n} represent the following integrals

$$A_{i+n} = \int_0^L m\left(\frac{x}{L}\right)^{i+n} dx \tag{21}$$

$$B_{i+n} = \int_0^L me\left(\frac{x}{L}\right)^{i+n} dx \tag{22}$$

$$C_{i+n} = \frac{i(i-1)n(n-1)}{L^4} \int_0^L EI\left(\frac{x}{L}\right)^{i+n-4} dx \tag{23}$$

$$D_{i+n} = \int_0^L I_m\left(\frac{x}{L}\right)^{i+n} dx \tag{24}$$

$$E_{i+n} = \frac{in}{L^2} \int_0^L GJ\left(\frac{x}{L}\right)^{i+n-2} dx \tag{25}$$

Different limits, s and t, are used on the summations for y and ϕ because the number of terms taken in the two expansions need not be the same. Retaining only the terms involving the coefficients a_0 through a_3 and b_0 through b_2, the two sets of characteristic equations may be written in the determinant forms shown in tables I and II. In the tables it is seen that terms involving the angle of sweep Λ appear only in one column and row of each determinant. When $\Lambda = 0$, these determinants lead to solutions for the free symmetrical and antisymmetrical vibrations of the unswept wing.

SOLUTION OF CHARACTERISTIC EQUATIONS

Values of a_n and b_n other than zero which satisfy the equations in tables I and II can be found only when the determinants of the systems of equations are zero. The determinants contain the unknown frequency ω; the values of ω which cause the determinants to be zero are the natural frequencies of vibration. To determine the mode associated with a given frequency, one of the unknown coefficients, a_n or b_n, is set equal to unity and any one of the equations is discarded. The resulting set of nonhomogeneous equations is then solved simultaneously to obtain the relative values of the other coefficients. With the coefficients known, the mode is obtained directly from equations (6) and (7).

The values of ω satisfying the frequency determinant may be found by several methods. Perhaps the simplest way to locate a frequency root is to evaluate the determinant for a number of trial values of ω in the expected vicinity of a natural frequency and to plot a curve of ω versus the value of the determinant. In most cases, the value of ω giving a zero determinant can be obtained from the results of three or four evaluations. The evaluations may be performed by the Crout method of solving determinants. (See reference 1.) The Crout method yields solutions rapidly and it provides for a running check which minimizes the possibility of computational error. With the procedure just outlined, any desired frequency root and mode can be found independently of the other frequencies and modes.

In the Crout solutions of the determinants presented in this paper, the calculations should be carried to at least eight significant figures. If an insufficient number of significant figures are carried, errors due to small differences of large numbers will cause difficulty in obtaining satisfactory check columns in the Crout solutions. The physical constants A_{i+n}, B_{i+n}, and so forth, in the equations, however, need not be computed to the number of significant figures used in solving the determinants. They merely need be computed as accurately as desired and then treated in the Crout solutions as being of absolute accuracy.

COMPUTATION OF CONSTANTS

A certain amount of preparatory calculation must be done before the characteristic equations can be solved for the frequencies and modes. This calculation consists of determining the constants A_{i+n}, B_{i+n}, C_{i+n}, and so forth. To evaluate the constants, the physical properties, m, I, I_m, J, e, E, and G, of the wing must be known at a number of stations x/L along the wing. Also necessary are the numerical values of the quantities $\left(\frac{x}{L}\right)^j$ which arise from the use of power series. For convenience, $\left(\frac{x}{L}\right)^j$ has been computed at 10 stations $\left(\frac{x}{L} = 0.1, 0.2, \ldots 0.9, 1.0\right)$ for j varying from 1 to 10. These data are presented in table III. The constants A_{i+n}, B_{i+n}, C_{i+n}, and so forth, are then found by multiplying the physical constants m, I, I_m, and so forth, by $\left(\frac{x}{L}\right)^j$ at each station along the wing and integrating over the span.

The integrals can be evaluated conveniently by use of the following numerical integration formula which is derived from the properties of a fifth degree curve.

$$\text{Area} = \frac{125\lambda}{144}(0.38a + 1.50b + 1.00c + 1.00d + 1.50e + 0.38f) \qquad (26)$$

In this equation, a, b, c, and so forth, are the ordinates at successive stations 0, 1, 2, and so forth, dividing the curve to be integrated into five equal sections a distance λ in length. For 10 sections, the formula is

$$\text{Area} = \frac{125\lambda}{144}(0.38a + 1.50b + 1.00c + 1.00d + 1.50e + 0.76f$$
$$+ 1.50g + 1.00h + 1.00i + 1.50j + 0.38k) \qquad (27)$$

Making use of this integration formula, a convenient procedure for calculating the constants A_{i+n}, B_{i+n}, C_{i+n}, and so forth, is:

(1) Divide the wing into 10 equal sections (a multiple of five).

(2) Tabulate the wing parameters m, EI, GJ, me, and I_m at each station (root station is 0 and tip station is 10) and multiply the tabulated parameter at station 0 by 0.38, at station 1 by 1.50, at station 2 by 1.00, and so forth, until the parameters at each station have all been "weighted" with the proper constant in equation (27). (In equation (27), a, b, c, and so forth, are actually the parameters multiplied by the valuos $\left(\frac{x}{L}\right)^j$; the work is simplified, however, by first multiplying the parameter by the "weighting factors" 0.38, 1.50, 1.00, and so forth, and then multiplying by $\left(\frac{x}{L}\right)^j$.)

(3) Multiply the "weighted" parametersat each station by the appropriate values of $\left(\frac{x}{L}\right)^j$ taken from table III.

(4) Add the products formed in (3) over the length of the wing and multiply the sums by $125\lambda/144$.

(5) Add to the sums in (4) the effect of concentrated masses which have not been included in the numerical integration. For instance, the addition to the constant A_{i+n} due to a concentrated mass, M, located at $\frac{x}{L} = 0.5$ would be $\left[M(0.5)^{i+n}\right]$.

ACCURACY OF RESULTS

Any analytical solution for airplane wing modes and frequencies must necessarily be based on simplifying assumptions, and the effect that these assumptions may have on the accuracy of the solution can only be determined by comparison of computed modes and frequencies with those determined experimentally. In the absence of experimental results, it is helpful to know, nevertheless, the degree to which the results from an energy solution check the results of an exact solution (based on the same simplifying assumptions). For comparison, an exact solution has been made for a specific example of a uniform wing mounted on a fuselage. The physical parameters of this system are shown in figure 2.

An energy solution for this case was made, assuming fourth degree power series for the expansion of both the deflection and rotation. Because the wing is uniform, the constants in the equations were determined by exact integration over the semispan. The resulting determinant for symmetrical vibration is presented in table IV. The zeros in the upper right and lower left quadrants of this determinant are due to the fact that there is no mass coupling along the wing. All the equations have been divided through by the factor mL, hence the appearance of constants such as 1/3, 1/4, 1/5, and so forth, and the ratio $R_M = \frac{m_F}{mL}$. The two lowest frequencies satisfying this determinant are compared with the exact frequencies in the table below.

Mode	Frequency (radians/sec)	
	Exact	Energy
1	54.3	54.3
2	157.4	157.4

The frequencies obtained in the exact and energy solutions were not determined to more significant figures. The results indicate, however, that the energy solution gives good accuracy. The modes associated with these frequencies are presented in figure 3. It is most probable that, for these two modes, a solution using third degree power series would have given satisfactory agreement with the exact solution.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va.

APPENDIX

DERIVATION OF EXPRESSIONS FOR KINETIC ENERGY

In the energy solution used in this paper, the potential energy stored in the wing at maximum displacement and the kinetic energy of the wing-fuselage system when passing through the equilibrium position must be known. This section gives the derivation of the kinetic-energy expression; the equations for potential energy of bending and twist are well known.

In figure 4, a cross section of the wing is shown at the instant it passes the equilibrium position; the elastic axis is assumed to have a vertical velocity v, and the cross section is assumed to be rotating at an angular velocity Ω. Any element of mass dm having the coordinate v,θ can be shown to have a total velocity such that

$$v_t^2 = (v + \Omega r \cos \theta)^2 + \Omega^2 r^2 \sin^2\theta \qquad \text{(A1)}$$

The kinetic energy of the element will be $\frac{1}{2}\,dm v_t^2$. If y and ϕ are the maximum values of deflection and rotation, the velocity v and rotational velocity Ω may be shown to be equal to ωy and $\omega\phi$, respectively. Substitution of these values in the expression for total velocity and integration of the kinetic energy of all the elements over the cross section gives for the total kinetic energy of unit length of the wing at the cross section under consideration

$$\frac{\omega^2}{2} m\left(y^2 + 2ey\phi + k^2\phi^2\right)$$

where e is the distance between the elastic axes and the center of gravity (e is positive when center of gravity is forward of the elastic axis) of the cross section and k is the radius of gyration of the cross section about the elastic axis. Integration of the kinetic energy over the length of the wing gives for the total kinetic energy of the wing

$$V_1 = \frac{\omega^2}{2}\int_0^L m\left(y^2 + 2ey\phi + k^2\phi^2\right) dx \qquad \text{(A2)}$$

The expression for the kinetic energy of vertical and pitching motion of the fuselage (half-fuselage) can be found by applying equation (A2) to the fuselage mass. The angle ϕ, however, is replaced by the pitching angle of the fuselage given by $\left(\phi \cos \Lambda - \frac{dy}{dx} \sin \Lambda\right)_{\frac{x}{L}=0}$. If Θ is used to denote the pitching angle, the kinetic energy of vertical and pitching motion of the fuselage is

$$\frac{\omega^2}{2}\left[m_F\left(y^2 + 2e_F y\Theta + k_F{}^2\Theta^2\right)\right]_{\frac{x}{L}=0}$$

or

$$\frac{\omega^2}{2}\left(m_F y^2 + 2m_F e_F y\Theta + I_{FP}\Theta^2\right)_{\frac{x}{L}=0}$$

The kinetic energy of the fuselage in rolling motion is

$$\frac{\omega^2}{2} I_{FR}\psi^2$$

where ψ is the angle of roll of the fuselage given by $\left(\phi \sin \Lambda + \frac{dy}{dx} \cos \Lambda\right)_{\frac{x}{L}=0}$. The total kinetic energy of the fuselage is then

$$V_2 = \frac{\omega^2}{2}\left(m_F y^2 + 2m_F e_F y\Theta + I_{FP}\Theta^2 + I_{FR}\psi^2\right)_{\frac{x}{L}=0} \qquad \text{(A3)}$$

REFERENCE

1. Crout, Prescott D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. Supp. to Elec. Eng., Trans. Section, AIEE, vol. 60, Dec. 1941, pp. 1235-1240. (Abridged as Marchant Methods MM-182, Sept. 1941, Marchant Calculating Machine Co., Oakland, Calif.)

TABLE I – SYSTEM OF EQUATIONS FOR SYMMETRICAL MODES AND FREQUENCIES OF SWEPT WING

	a_0	a_2	a_3	b_0	b_1	b_2
For a_i						
$i = 0$	$\left(A_0 + m_F\right)$	A_2	A_3	$\left(B_0 + \frac{m_F e_F}{\cos \Lambda} - LA_1 \tan \Lambda\right)$	B_1	B_2
$i = 2$	A_2	$\left(A_4 - \frac{C_4}{\omega^2}\right)$	$\left(A_5 - \frac{C_5}{\omega^2}\right)$	$\left(B_2 - LA_3 \tan \Lambda\right)$	B_3	B_4
$i = 3$	A_3	$\left(A_5 - \frac{C_5}{\omega^2}\right)$	$\left(A_6 - \frac{C_6}{\omega^2}\right)$	$\left(B_3 - LA_4 \tan \Lambda\right)$	B_4	B_5
For b_i						
$i = 0$	$\left(B_0 + \frac{m_F e_F}{\cos \Lambda} - LA_1 \tan \Lambda\right)$	$\left(B_2 - LA_3 \tan \Lambda\right)$	$\left(B_3 - LA_4 \tan \Lambda\right)$	$\left(D_0 + L^2 A_2 \tan^2 \Lambda + \frac{I_{FP}}{\cos^2 \Lambda} - 2LB_1 \tan \Lambda\right)$	$\left(D_1 - LB_2 \tan \Lambda\right)$	$\left(D_2 - LB_3 \tan \Lambda\right)$
$i = 1$	B_1	B_3	B_4	$\left(D_1 - LB_2 \tan \Lambda\right)$	$\left(D_2 - \frac{E_2}{\omega^2}\right)$	$\left(D_3 - \frac{E_3}{\omega^2}\right)$
$i = 2$	B_2	B_4	B_5	$\left(D_2 - LB_3 \tan \Lambda\right)$	$\left(D_3 - \frac{E_3}{\omega^2}\right)$	$\left(D_4 - \frac{E_4}{\omega^2}\right)$

$= 0$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE II – SYSTEM OF EQUATIONS FOR ANTISYMMETRICAL MODES AND FREQUENCIES OF SWEPT WING

$$
\begin{array}{ll|ccccc|l}
 & & a_1 & a_2 & a_3 & b_1 & b_2 & \\
\text{For} & a_i & & & & & & \\
 & i = 1 & \left(A_2 + \frac{1}{L^2}D_0 \tan^2\Lambda + \frac{I_{FR}}{L^2 \cos^2\Lambda} + \frac{2}{L}B_1 \tan\Lambda\right) & \left(A_3 + \frac{1}{L}B_2 \tan\Lambda\right) & \left(A_4 + \frac{1}{L}B_3 \tan\Lambda\right) & \left(B_2 + \frac{1}{L}D_1 \tan\Lambda\right) & \left(B_3 + \frac{1}{L}D_2 \tan\Lambda\right) & \\
 & i = 2 & \left(A_3 + \frac{1}{L}B_2 \tan\Lambda\right) & \left(A_4 - \frac{C_4}{\omega^2}\right) & \left(A_5 - \frac{C_5}{\omega^2}\right) & B_3 & B_4 & \\
 & i = 3 & \left(A_4 + \frac{1}{L}B_3 \tan\Lambda\right) & \left(A_5 - \frac{C_5}{\omega^2}\right) & \left(A_6 - \frac{C_6}{\omega^2}\right) & B_4 & B_5 & = 0 \\
\text{For} & b_i & & & & & & \\
 & i = 1 & \left(B_2 + \frac{1}{L}D_1 \tan\Lambda\right) & B_3 & B_4 & \left(D_2 - \frac{E_2}{\omega^2}\right) & \left(D_3 - \frac{E_3}{\omega^2}\right) & \\
 & i = 2 & \left(B_3 + \frac{1}{L}D_2 \tan\Lambda\right) & B_4 & B_5 & \left(D_3 - \frac{E_3}{\omega^2}\right) & \left(D_4 - \frac{E_4}{\omega^2}\right) &
\end{array}
$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE III – VALUES OF $\left(\frac{x}{L}\right)^j$

$\left(\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^2$	$\left(\frac{x}{L}\right)^3$	$\left(\frac{x}{L}\right)^4$	$\left(\frac{x}{L}\right)^5$	$\left(\frac{x}{L}\right)^6$	$\left(\frac{x}{L}\right)^7$	$\left(\frac{x}{L}\right)^8$	$\left(\frac{x}{L}\right)^9$	$\left(\frac{x}{L}\right)^{10}$
0.1	0.01	0.001	0.0001	0.00001	0.000001	0.0000001	0.00000001	0.000000001	0.0000000001
.2	.04	.008	.0016	.00032	.000064	.0000128	.00000256	.000000512	.0000001024
.3	.09	.027	.0081	.00243	.000729	.0002187	.00006561	.000019683	.0000059049
.4	.16	.064	.0256	.01024	.004096	.0016384	.00065536	.000262144	.0001048576
.5	.25	.125	.0625	.03125	.015625	.0078125	.00390625	.001953125	.0009765625
.6	.36	.216	.1296	.07776	.046656	.0279936	.01679616	.010077696	.0060466176
.7	.49	.343	.2401	.16807	.117649	.0823543	.05764801	.040353607	.0282475249
.8	.64	.512	.4096	.32768	.262144	.2097152	.16777216	.134217728	.1073741824
.9	.81	.729	.6561	.59049	.531441	.4782969	.43046721	.387420489	.3486784401
1.0	1.00	1.000	1.0000	1.00000	1.000000	1.0000000	1.00000000	1.000000000	1.0000000000

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

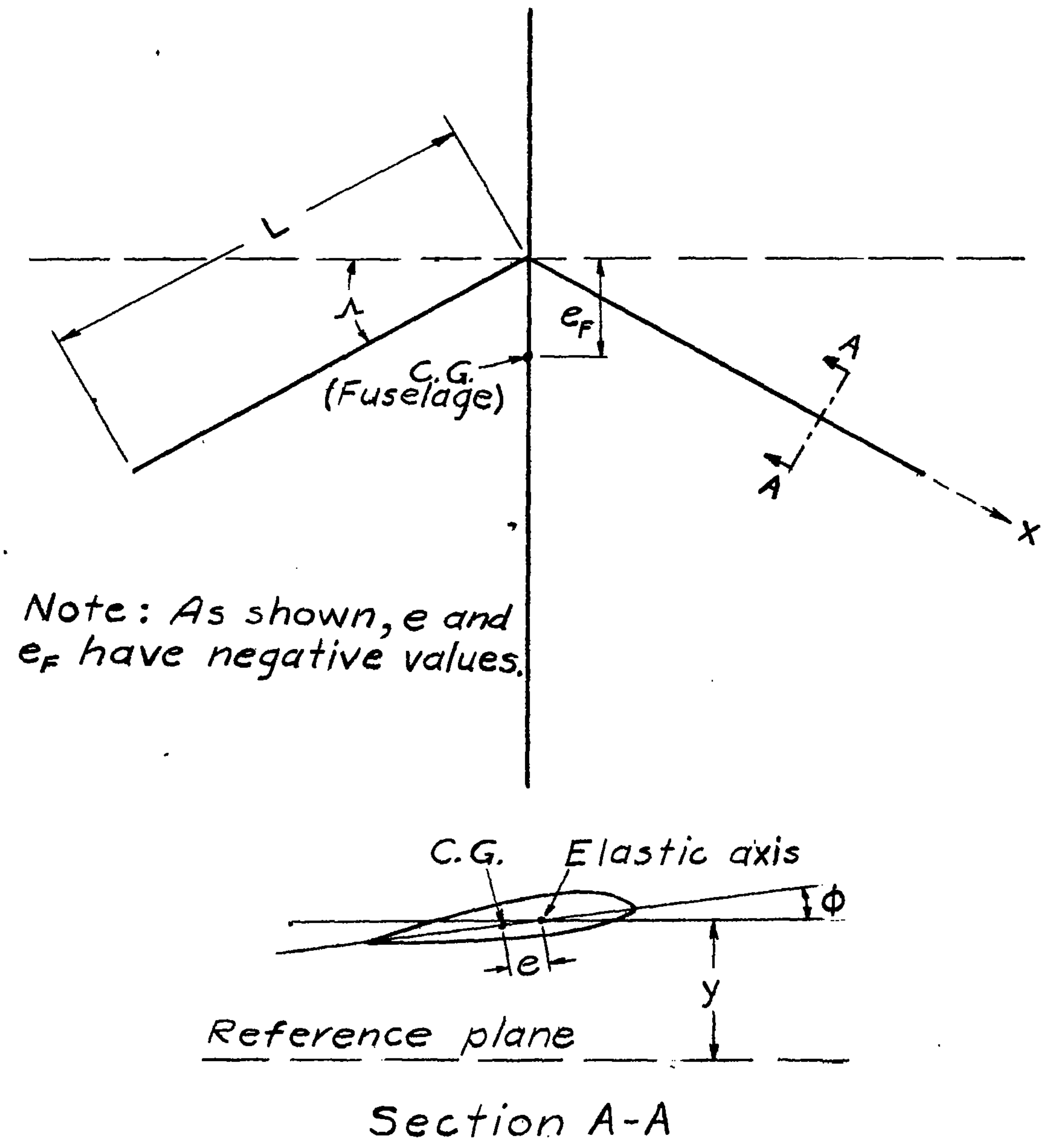
TABLE IV – SYSTEM OF EQUATIONS APPLICABLE TO NUMERICAL EXAMPLE

	a_0	a_2	a_3	a_4	b_0	b_1	b_2	b_3	b_4
For a_1									
$i = 0$	$\left(1 + R_m\right)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\left(\frac{e_F R_m}{\cos\Lambda} - \frac{L}{2}\tan\Lambda\right)$	0	0	0	0
$i = 2$	$\frac{1}{3}$	$\frac{1}{5} - \frac{4k_1}{\omega^2}$	$\frac{1}{6} - \frac{6k_1}{\omega^2}$	$\frac{1}{7} - \frac{8k_1}{\omega^2}$	$-\frac{L}{4}\tan\Lambda$	0	0	0	0
$i = 3$	$\frac{1}{4}$	$\frac{1}{6} - \frac{6k_1}{\omega^2}$	$\frac{1}{7} - \frac{12k_1}{\omega^2}$	$\frac{1}{8} - \frac{18k_1}{\omega^2}$	$-\frac{L}{5}\tan\Lambda$	0	0	0	0
$i = 4$	$\frac{1}{5}$	$\frac{1}{7} - \frac{8k_1}{\omega^2}$	$\frac{1}{8} - \frac{18k_1}{\omega^2}$	$\frac{1}{9} - \frac{144k_1}{5\omega^2}$	$-\frac{L}{6}\tan\Lambda$	0	0	0	0
For b_1									
$i = 0$	$\left(\frac{e_F R_m}{\cos\Lambda} - \frac{L}{2}\tan\Lambda\right)$	$-\frac{L}{4}\tan\Lambda$	$-\frac{L}{5}\tan\Lambda$	$-\frac{L}{6}\tan\Lambda$	$\left(\frac{L^2}{3}\tan^2\Lambda + \frac{I_m}{m} + \frac{I_{mp}}{mL\cos^2\Lambda}\right)$	$\frac{I_m}{2m}$	$\frac{I_m}{3m}$	$\frac{I_m}{4m}$	$\frac{I_m}{5m}$
$i = 1$	0	0	0	0	$\frac{I_m}{2m}$	$\frac{I_m}{3m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{4m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{5m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{6m} - \frac{k_2}{\omega^2}$
$i = 2$	0	0	0	0	$\frac{I_m}{3m}$	$\frac{I_m}{4m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{5m} - \frac{4k_2}{3\omega^2}$	$\frac{I_m}{6m} - \frac{3k_2}{2\omega^2}$	$\frac{I_m}{7m} - \frac{8k_2}{5\omega^2}$
$i = 3$	0	0	0	0	$\frac{I_m}{4m}$	$\frac{I_m}{5m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{6m} - \frac{3k_2}{2\omega^2}$	$\frac{I_m}{7m} - \frac{9k_2}{5\omega^2}$	$\frac{I_m}{8m} - \frac{2k_2}{\omega^2}$
$i = 4$	0	0	0	0	$\frac{I_m}{5m}$	$\frac{I_m}{6m} - \frac{k_2}{\omega^2}$	$\frac{I_m}{7m} - \frac{8k_2}{5\omega^2}$	$\frac{I_m}{8m} - \frac{2k_2}{\omega^2}$	$\frac{I_m}{9m} - \frac{16k_2}{7\omega^2}$

$= 0$

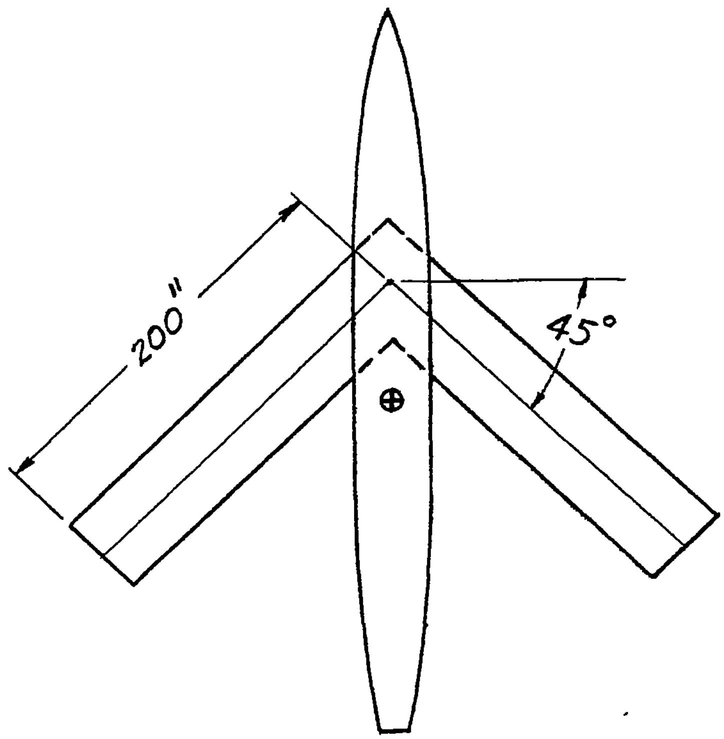
$k_1 = \frac{EI}{mL^4}$ $\qquad k_2 = \frac{GJ}{mL^2}$ $\qquad R_m = \frac{m_F}{mL}$ $\qquad a_1 = -b_0 L \tan\Lambda$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS



NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 1.- Coordinate system used in analysis.



Wing Parameters

$E = 10{,}000{,}000$ psi $\quad G = 4{,}000{,}000$ psi

$I = 800$ in.^{4} $\quad J = 1600$ in.^{4}

$m = 0.025$ lb-sec^2/in.^{2} $\quad I_m = 16$ lb-sec^2

Fuselage Parameters

$I_{FP} = 400{,}000$ lb.-sec^2-in. $\quad I_{FR} = 10{,}000$ lb-sec^2-in.

$e_F = -70$ in. $\quad R_m = \frac{m_F}{mL} = 3$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 2.- Parameters for numerical example.

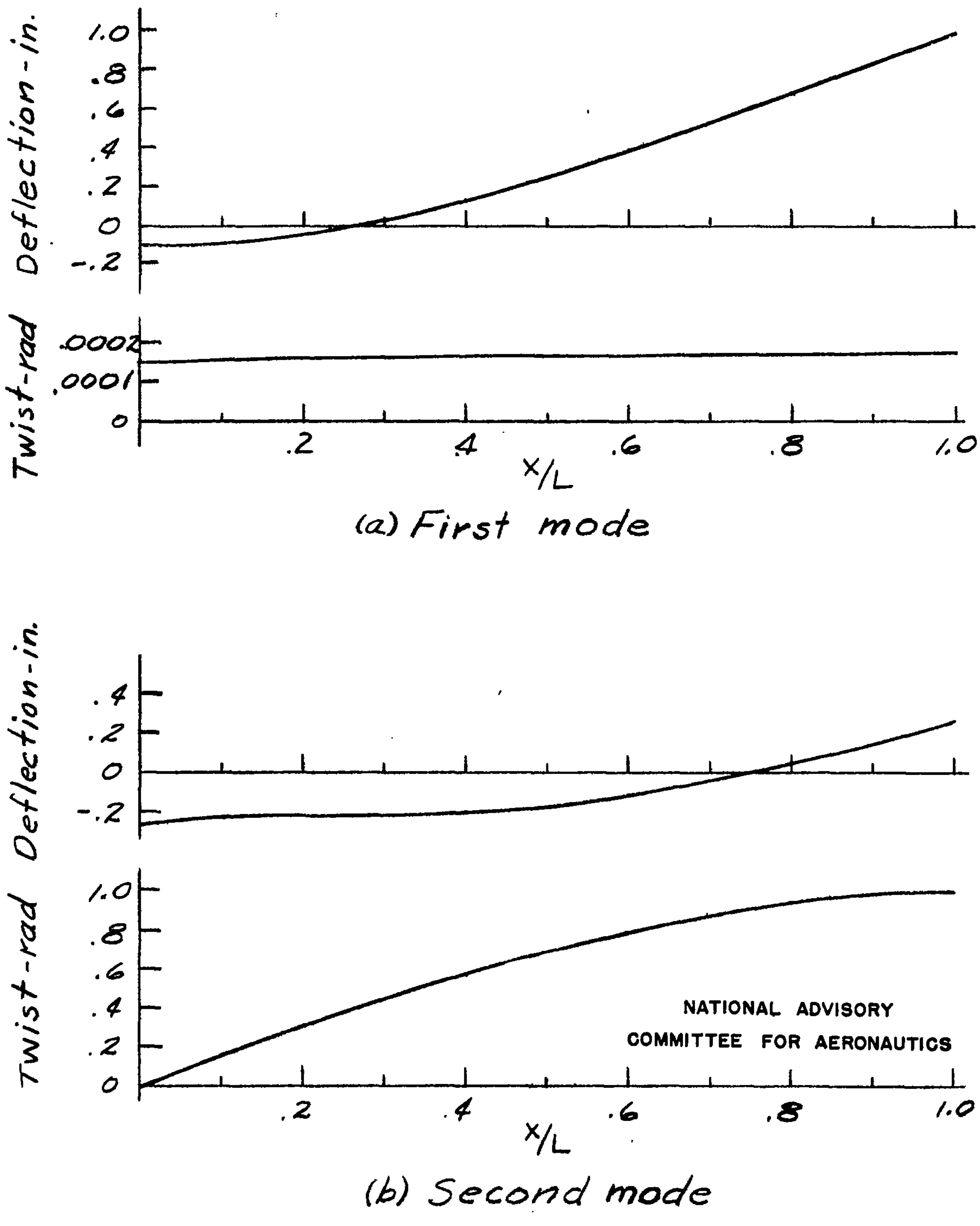
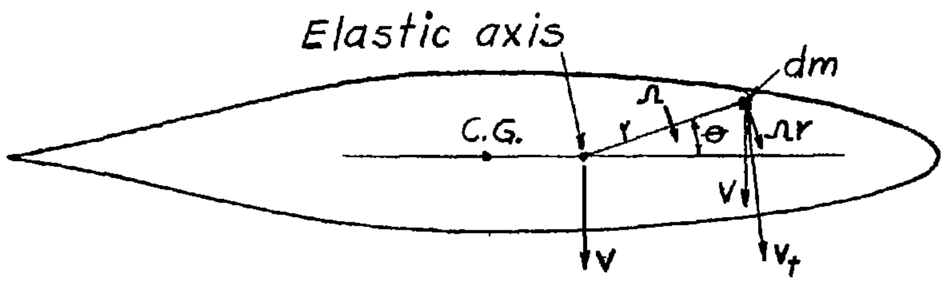


Figure 3. - Symmetrical modes of vibration.



NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 4.– Coordinate and velocity notation for element of mass on cross section.